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METHODS OF STABILIZING THE MOTIONS OF REVERSIBLE DYNAMIC SYSTEMS USING NON-LINEAR CANONICAL TRANSFORMATIONS†

Yu. K. ZOTOV and A. V. TIMOFEYEV

St Petersburg

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Methods of synthesizing stabilizing and robust control laws for non-linear reversible systems which ensure asymptotic stability of programmed motions, specified figures of merit and decomposition of transients are considered. Non-linear canonical transformations of state space and the controls are obtained which simplify the synthesis and analysis of the laws of the stabilization of reversible dynamic systems.

1. FORMULATION OF THE PROBLEM

Consider a controlled system, the dynamics of which is described by a system of ordinary differential equations of the form

$$z^{\cdot} = F(z,u,t), \quad z(t_0) = z_0, \quad t \geqslant t_0 \tag{1.1}$$

Here z_0, $z = z(t)$ are n-dimensional vectors of the states of the system at the initial and present instants of time, u is an m-dimensional vector of the controls, and F is an n-dimensional vector function which satisfies the conditions for a solution of system (1.1) to exist and to be unique.

Suppose we are given a programmed motion $z_p = z_p(t)$, $t \geqslant t_0$ which is a particular solution of system (1.1) for a certain permissible programmed control $u = u_p = u_p(t)$ and initial condition $z_{p0} = z_p(t_0)$. The programmed motion $z_p(t)$ will be called the unperturbed motion, while any other motion $z(t)$ of system (1.1) under the action of acceptable controls will be called a perturbed (real) motion. Then the quantities

$$e_z = z - z_p, \quad e_u = u - u_p \tag{1.2}$$

are perturbations, i.e. deviations of the real (perturbed) and programmed motions, which satisfy the equation in deviations

$$e_z^{\cdot} = F_z(e_z, e_u, t), \quad e_z(t_0) = e_{z_0}, \quad t \geqslant t_0 \tag{1.3}$$

Here

$$F_z(e_z, e_u, t) = F_z(e_z + z_p, e_u + u_p, t) - F(z_p, u_p, t) \tag{1.4}$$

where $F_z(0, 0, t) \equiv 0$.

For a wide range of dynamical systems the structure of Eqs (1.3) and (1.4) is such that

$$e_z = \operatorname{col}(e_{z_1}, \ldots, e_{z_r}), \quad n = mr \tag{1.5}$$

$$F_z(e_z, e_u, t) = \operatorname{col}(F_{z_1}(e_z^2, t), \ldots, F_{z_{r-1}}(e_z^r, t), F_{z_r}(e_z^r, e_u, t)) \tag{1.6}$$

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$$F_{z_i}(e_z^{i+1}, t) = C_i(e_z^i, t) + D_i(e_z^i, t)e_{z_{i+1}}, \quad i = 1, \ldots, r-1 \tag{1.7}$$

$$F_{z_r}(e_z^r, e_u, t) = C_r(e_z^r, t) + D_r(e_z^r, t)e_u \tag{1.8}$$

Here e_{z_i} is an m-dimensional vector, $e_z^i = \mathrm{col}(e_{z_1}, \ldots, e_{z_i})$ are mi-dimensional vectors, C_i and D_i ($i = 1, \ldots, r$) are specified vector and matrix functions, the vector functions F_{z_i} ($i = 1, \ldots, r$) (1.7) and (1.8) are continuous and continuously differentiable a sufficient number of times with respect to their arguments, and the matrix functions D_i ($i = 1, \ldots, r$) in (1.7) and (1.8) are non-degenerate for all possible values of their arguments, i.e.

$$\mathrm{rank}\, D_i(e_z^i, t) = m, \quad \forall e_z^i \in R^{mi}, \quad t \geqslant t_0, \quad i = 1, \ldots, r \tag{1.9}$$

where R^{mi} is mi-dimensional Euclidean space.

Examples of such systems are mechanical and electromechanical systems described by the Lagrange–Maxwell equations.

Equations (1.3)–(1.9) can be written in an expanded form with respect to the control e_u

$$\dot{e}_{z_i} = F_{z_i}(e_z^{i+1}, t) = C_i(e_z^i, t) + D_i(e_z^i, t)e_{z_{i+1}}, \quad i = 1, \ldots, r-1 \tag{1.10}$$

$$e_u = D_r^{-1}(e_z^r, t)(\dot{e}_{z_r} - C_r(e_z^r, t)) \tag{1.11}$$

System (1.3)–(1.9), which possesses the above-mentioned property of solvability, belongs to the class of reversible controlled systems. It follows from the fact that it is reversible, taking (1.2) into account, that the initial system (1.1), (1.4)–(1.9) is also reversible. It was shown in [1–6] that the reversible controlled system (1.1), (1.4)–(1.9) possesses the property of global controllability and it is easy to construct a programmed motion $z_p(t)$ for it in analytic form and the corresponding programmed control $u_p(t)$.

We will assume that the following relations hold for each of the vector functions C_i ($i = 1, \ldots, r$) and the matrix functions D_i ($i = 1, \ldots, r$) for all possible values of their arguments

$$|C_i(e_z^i, t)| \leqslant \sum_{j=1}^{k_i} a_{ij}|e_z^i|^j, \quad \forall e_z^i \in R^{mi}, \quad t \geqslant t_0, \quad i = 1, \ldots, r \tag{1.12}$$

$$|D_i(e_z^i, t)| \leqslant d_i < \infty, \quad \forall e_z^i \in R^{mi}, \quad t \geqslant t_0, \quad i = 1, \ldots, r \tag{1.13}$$

where $a_{ij} \geqslant 0$ ($j = 1, \ldots, k_i$), $d_i > 0$ are certain constants. We will assume that similar relations hold for the partial derivatives of C_i ($i = 1, \ldots, r$) and D_i ($i = 1, \ldots, r$) with respect to their arguments.

We will say that the programmed motion $z_p(t)$ of system (1.1) is stabilized if a control law exists with feedback with respect to the state vector $z(t)$ of the form

$$u = u(z, t) = u_p(t) + e_u(z - z_p, t) \tag{1.14}$$

which ensures the asymptotic stability of $z_p(t)$ of system (1.1) and (1.4)–(1.9) as a whole.

The properties, criteria and laws of stabilization of the programmed motion of reversible controlled systems further develop and extend the results obtained previously in [1–14].

2. CANONICAL FORMS OF THE DESCRIPTION OF REVERSIBLE CONTROLLED SYSTEMS

The proposed method of analysing the stability and of synthesizing stabilizing controls for non-linear reversible controlled systems are based on reducing (1.1) and (1.4)–(1.9) to certain simple "canonical forms" using non-linear transformations of the coordinates in state and control spaces. It is important to note that for reversible controlled systems in canonical form the problems involved in analysing the stability of the programmed motion and of synthesizing stabilizing controls are simplified considerably.

The canonical form of the first type is the representation of the reversible controlled system in the form

$$\dot{e}_x = P(e_x^r,t)e_x + Q(e_x^r,t)e_w, \quad e_x(t_0) = e_{x0}, \quad t \geq t_0 \tag{2.1}$$

where

$$e_x = \mathrm{col}(e_{x_1},\ldots,e_{x_r}) \tag{2.2}$$

is an n-dimensional vector of the "canonical" state of the system, e_{x_i}, $e_x^i = \mathrm{col}(e_{x_1}, \ldots, e_{x_i})$ are m- and mi-dimensional vectors, e_w is an m-dimensional vector of the "canonical" control, and P and Q are $n \times n$ and $n \times m$ matrix functions of the form

$$P(e_x^r,t) = \begin{Vmatrix} P_{11}(e_x^1,t) & P_{12}(e_x^1,t) & 0 & & \ldots 0 \\ P_{21}(e_x^2,t) & P_{22}(e_x^2,t) & P_{23}(e_x^2,t) & 0 & \ldots 0 \\ \vdots & & \ddots & \ddots & \vdots \\ P_{r-2,1}(e_x^{r-2},t) & \ldots & P_{r-2,r-1}(e_x^{r-2},t) & & 0 \\ P_{r-1,1}(e_x^{r-1},t) & \ldots & P_{r-1,r-1}(e_x^{r-1},t) & & P_{r-1,r}(e_x^{r-1},t) \\ P_{r-1}(e_x^r,t) & \ldots & P_{r,r-1}(e_x^r,t) & & P_{rr}(e_x^r,t) \end{Vmatrix} \tag{2.3}$$

$$Q(e_x^r,t) = \begin{Vmatrix} 0 \\ Q_r(e_x^r,t) \end{Vmatrix} \tag{2.4}$$

where $|P(e_x^r, t)| \leqslant \kappa_1 < \infty$, $|Q(e_x^r, t)| \leqslant \kappa_2 < \infty$, $\forall e_x^r \in R^n$, $t \geqslant t_0$, where κ_1, κ_2 are certain positive constants, and for the partial derivatives of the element-functions of the $m \times m$ blocks P_{ij} ($i = 1, \ldots, r$; $j = 1, \ldots, i$) of the matrix-function P with respect to their arguments, relations similar to (1.12) are satisfied, while for the partial derivatives of the element-functions of the $m \times m$ blocks $P_{i,j+1}$ ($i = 1, \ldots, r-1$) of the matrix M and the $m \times m$ block Q_r of the matrix Q with respect to their arguments, relations similar to (1.13) are satisfied, and O is the zero matrix of corresponding dimensions.

For reversible controlled systems, the matrix functions $P_{i,j+1}$ ($i = 1, \ldots, r-1$) and the block Q_r are non-degenerate for all possible values of their arguments, i.e.

$$\mathrm{rank}\, P_{i,i+1}(e_x^i,t) = m, \quad \forall e_x^i, t \in R^{mi}, \quad t \geqslant t_0, \quad i = 1,\ldots,r-1 \tag{2.5}$$

$$\mathrm{rank}\, Q_r(e_x^r,t) = m, \quad \forall e_x^r \in R^n, \quad t \geqslant t_0 \tag{2.6}$$

Within the framework of the canonical representation of reversible controlled systems of the first type it is best to distinguish two subclasses of canonical forms which are distinguished by the structure of the matrix function P in Eq. (2.1), namely

$$P(e_x^r,t) = \begin{Vmatrix} P_{11}(,t) & P_{12}(e_x^1,t) & 0 & & & & \ldots 0 \\ -P_{12}^*(e_x^1,t) & P_{22}(e_x^1,t) & P_{23}(e_x^2,t) & 0 & & & \ldots 0 \\ 0 & -P_{23}^*(e_x^2,t), & P_{33}(e_x^2,t) & \ddots & \ddots & & \vdots \\ \vdots \quad \ddots \quad \ddots & & \ddots & & & P_{r-2,r-1}(e_x^{r-2},t) & 0 \\ 0 \;\ldots\; 0 & & -P_{r-2,r-1}^*(e_x^{r-2},t) & & & P_{r-1,r-1}(e_x^{r-2},t) & P_{r-1,r}(e_x^{r-1},t) \\ 0 \;\ldots & & 0 & & & -P_{r-1,r}^*(e_x^{r-1},t) & P_{rr}(e_x^{r-1},t) \end{Vmatrix} \tag{2.7}$$

$$P(e_x^{r-1},t)=\begin{Vmatrix} P_{11}(t) & P_{12}(t) & 0 & & & \cdots & 0 \\ 0 & P_{22}(e_x^1,t) & P_{23}(e_x^1,t) & 0 & & \cdots & 0 \\ \vdots & \ddots & & \ddots & & \ddots & \vdots \\ 0 \cdots & & 0 & P_{r-2,r-2}(e_x^{r-3},t) & P_{r-2,r-1}(e_x^{r-3},t) & & 0 \\ 0 \cdots & & & 0 & P_{r-1,r-1}(e_x^{r-2},t) & & P_{r-1,r}(e_x^{r-2},t) \\ 0 \cdots & & & & 0 & & P_{rr}(e_x^{r-1},t) \end{Vmatrix} \tag{2.8}$$

The canonical form of the second type is the representation of the reversible controlled system in the form

$$\dot{e}_x = Pe_x + Qe_w, \quad e_x(t_0)=e_{x_0}, \quad t \geqslant t_0 \tag{2.9}$$

where e_x is the n-dimensional vector (2.2) of the "canonical state" of the system, and P and Q are constant matrices having the same structure as the matrix functions P (2.3) and Q (2.4), where the $m \times m$ blocks $P_{i,j+1}$ ($i = 1, \ldots, r-1$) of the matrix P and the $m \times m$ block Q_r of the matrix Q are non-degenerate, i.e.

$$\operatorname{rank} P_{i,i+1} = m, \quad i=1,\ldots,r-1; \quad \operatorname{rank} Q_r = m \tag{2.10}$$

Note that in the special case when the matrices P and Q in (2.9) have the form [4–6]

$$P=\begin{Vmatrix} 0 & I_{n-m} \\ 0 & 0 \end{Vmatrix}, \quad Q=\begin{Vmatrix} 0 \\ Q_r \end{Vmatrix} \tag{2.11}$$

where I_m is the unit $m \times m$ matrix, the reversible programmed control (2.9) has the simplest "canonical form", and its state vector is determined by canonical variables of the form

$$e_x=\operatorname{col}(e_{x_1},\ldots,e_{x_r})=\operatorname{col}(e_{x_1},\dot{e}_{x_1},\ldots,e_{x_1}^{(r-1)}), \quad e_{x_i}=\dot{e}_{x_{i-1}}, \quad i=2,\ldots,r \tag{2.12}$$

where $\overset{\dots}{e}_{x_1}$ is the ith derivative with respect to time t of the variable $e_{x_1} = e_{x_1}(t)$.

3. REDUCTION OF THE REVERSIBLE CONTROLLED SYSTEMS TO CANONICAL FORM

We will construct a one-to-one transformation of the state and control spaces of the initial reversible controlled system (1.3)–(1.9) which reduce it to a simpler "canonical form" (2.1)–(2.6).

We will seek transformation in the form

$$e_x=\Psi^r(e_z,t), \quad e_w=\Psi_{r+1}(e_z,e_u,t) \tag{3.1}$$

where Ψ^r and Ψ_{r+1} are n- and m-vector functions of the following respective forms

$$\Psi^r(e_z,t)=\operatorname{col}(\Psi_1(e_z^1,t),\ldots,\Psi_r(e_z^r,t)) \tag{3.2}$$

$$\Psi_1(e_z^1,t)=K_1+L_1e_{z_1}, \quad K_1=0, \quad L_1=I_m \tag{3.3}$$

$$\Psi_i(e_z^i,t)=K_i(e_z^{i-1},t)+L_i(e_z^{i-1},t)e_{z_i}, \quad i=2,\ldots,r \tag{3.4}$$

$$\Psi_{r+1}(e_z,e_u,t)=K_{r+1}(e_z,t)+L_{r+1}(e_z,t)e_u \tag{3.5}$$

Here K_i ($i = 2, \ldots, r+1$) is an m-vector function, and L_i ($i = 2, \ldots, r+1$) are $m \times m$ matrix functions, to be determined.

We will write an algorithm for obtaining the unknown vector-functions K_i ($i = 2, \ldots, r+1$) and the matrix-functions L_i ($i = 2, \ldots, r+1$).

Consider the r identities

$$\begin{aligned} e_{x_1}^{\cdot} &= \Psi_1^{\cdot}(e_z^1,t) = K_1^{\cdot} + L_1^{\cdot} e_{z_1} + L_1 e_{z_1}^{\cdot} = e_{z_1}^{\cdot} \\ e_{x_i}^{\cdot} &= \Psi_i^{\cdot}(e_z^1,t) = K_i^{\cdot}(e_z^{i-1},t) + L_i^{\cdot}(e_z^{i-1})e_{z_i} + L_i(e_z^{i-1},t)e_{z_i}^{\cdot}, \quad i=2,\ldots,r-1 \\ e_{x_r}^{\cdot} &= \Psi_r^{\cdot}(e_z^r,t) = K_r^{\cdot}(e_z^{r-1},t) + L_r^{\cdot}(e_z^{r-1},t)e_{z_r} + L_r(e_z^{r-1},t)e_{z_r}^{\cdot} \end{aligned} \tag{3.6}$$

Replacing the derivatives $e_{x_i}^{\cdot}$ ($i = 1, \ldots, r$) and $e_{z_i}^{\cdot}$ ($i = 1, \ldots, r$) in (3.6) by means of the formulae

$$\begin{aligned} e_{x_i}^{\cdot} &= F_{x_i}(e_x^{i+1},t) = P_{i1}(e_x^i,t)e_{x_1} + \ldots + P_{i,i+1}(e_x^i,t)e_{x_{i+1}}, \quad i=1,\ldots,r-1 \\ e_{x_r}^{\cdot} &= F_{x_r}(e_x^r,e_w,t) = P_{r1}(e_x^r,t) + \ldots + P_{rr}(e_x^r,t)e_{x_r} + Q_r(e_x^r,t)e_w \end{aligned} \tag{3.7}$$

and using (1.7) and (1.8), we obtain the relations

$$\begin{aligned} & P_{11}(e_x^1,t)e_{x_1} + P_{12}(e_x^1,t)e_{x_2} = K_1^{\cdot} + L_1^{\cdot} e_{z_1} + L_1(C_1(e_z^1,t) + D_1(e_z^1,t)e_{z_2}) \\ & P_{i1}(e_x^i,t)e_{x_1} + \ldots + P_{i,i+1}(e_x^i,t)e_{x_{i+1}} = K_i^{\cdot}(e_z^{i-1},t) + L_i^{\cdot}(e_z^{i-1},t)e_{z_i} + \\ & + L_i(e_z^{i-1},t)(C_i(e_z^i,t) + D_i(e_z^i,t)e_{z_{i+1}}), \quad i=2,\ldots,r-1 \\ & P_{r1}(e_x^r,t)e_{x_1} + \ldots + P_{rr}(e_x^r,t)e_{x_r} + Q_r(e_x^r,t)e_w = K_r^{\cdot}(e_z^{r-1},t) + \\ & + L_r^{\cdot}(e_z^{r-1},t)e_{z_r} + L_r(e_z^{r-1},t)(C_r(e_z,t) + D_r(e_z,t)e_u) \end{aligned} \tag{3.8}$$

In a reversible controlled system of canonical form (2.1)–(2.6) the matrix-function P_{12} is non-degenerate by virtue of (2.5). Hence, the first equation (with $i = 1$) of system (3.8) can be solved for e_{x_2} and we can obtain the desired second transformation (with $i = 2$) from (3.4) connecting the variables e_{x_2} and e_{z_2} with vector function $\Psi_2(e_z^2, t)$, in which

$$\begin{aligned} K_2(e_z^1,t) &= P_{12}^{-1}(e_z^1,t)[K_1^{\cdot} + L_1^{\cdot} e_{z_1} + L_1 C_1(e_z^1,t) - P_{11}(e_z^1,t)e_{z_1}] = P_{12}^{-1}(e_z^1,t)[C_1(e_z^1,t) - P_{11}(e_z^1,t)e_{z_1}] \\ L_2(e_z^1,t) &= P_{12}^{-1}(e_z^1,t)D_1(e_z^1,t) \end{aligned} \tag{3.9}$$

Continuing this process, i.e. substituting into each current ith equation (beginning with $i = 2$) from (3.8) the previously obtained vector-functions Ψ_j ($j = 1, \ldots, i$), K_i and the matrix function L_i and taking into account, by (2.5) and (2.6), the fact that the matrix functions $P_{i,i+1}$ ($i = 2, \ldots, r-1$) and Q_r are non-degenerate, we obtain the required transformations of (3.4) and (3.5) (for $i = 3, \ldots, r+1$), in which

$$\begin{aligned} & K_i(e_z^{i-1},t) = P_{i-1,i}^{-1}(\Psi^{i-1},t)[K_{i-1}^{\cdot}(e_z^{i-2},t) + L_{i-1}^{\cdot}(e_z^{i-2},t)e_{z_{i-1}} + \\ & + L_{i-1}(e_z^{i-2},t)C_{i-1}(e_z^{i-1},t) - \sum_{k=1}^{i-1} P_{i-1,k}(\Psi^{i-1},t)\Psi_k], \quad i=3,\ldots,r \\ & L_i(e_z^{i-1},t) = P_{i-1,i}^{-1}(\Psi^{i-1},t)L_{i-1}(e_z^{i-2},t)D_{i-1}(e_z^{i-1},t), \quad i=3,\ldots,r \\ & K_{r+1}(e_z,t) = Q_r^{-1}(\Psi^r,t)[K_r^{\cdot}(e_z^{r-1},t) + L_r^{\cdot}(e_z^{r-1},t)e_{z_r} + \\ & + L_r(e_z^{r-1},t)C_r(e_z^r,t) - \sum_{k=1}^{r} P_{rk}(\Psi^r,t)\Psi_k] \\ & L_{r+1}(e_z,t) = Q_r^{-1}(\Psi^r,t)L_r(e_z^{r-1},t)D_r(e_z^r,t) \\ & \Psi^k = \operatorname{col}(\Psi_1,\ldots,\Psi_k), \quad \Psi^k = \Psi^k(e_z^k,t), \\ & \Psi_k = \Psi_k(e_z^k,t), \quad k=1,\ldots,i-1; \quad \Psi^r = \Psi^r(e_z^r,t) \end{aligned} \tag{3.10}$$

Since, taking relations (3.3), (3.4), (3.10), (1.9), (2.5) and (2.6) into account we have

$$\operatorname{rank} L_l = m, \quad \operatorname{rank} L_i(e_z^{i-1}, t) = m, \quad \forall e_z^{i-1} \in R^{m(i-1)}, \quad t \geqslant t_0, \quad i = 2, \ldots, r+1 \tag{3.11}$$

and the $n \times m$ Jacobi matrix $J_{\Psi^r}(e_z^{r-1}, t) = \partial\Psi^r(e_z, t)/\partial e_z$ has a block lower-triangular form, we have

$$\operatorname{rank} J_{\Psi^r}(e_z^{r-1}, t) = n, \quad \forall e_z^{r-1} \in R^{m(i-1)}, \quad t \geqslant t_0 \tag{3.12}$$

Taking (3.1), (3.5) and (3.11) into account we have

$$\begin{gathered}\operatorname{rank} J_{\Psi_{r+1}}(e_z, t) = m, \quad \forall e_z \in R^n, \quad t \geqslant t_0 \\ J_{\Psi_{r+1}}(e_z, t) = \partial\Psi_{r+1}(e_z, e_u, t)/\partial e_u = L_{r+1}(e_z, t)\end{gathered} \tag{3.13}$$

Solving the first equation of (3.1) for e_z and the second equation for e_u, we obtain the inverse transformations

$$\begin{gathered}e_z = \Phi^r(e_x, t) = \operatorname{col}(\Phi_1(e_x^1, t), \ldots, \Phi_r(e_x^r, t)) \\ e_u = \Phi_{r+1}(e_x, e_w, t)\end{gathered} \tag{3.14}$$

where

$$\begin{gathered}\Phi_1(e_x^1, t) = M_1 + N_1 e_{x_1}, \quad \Phi_i(e_x^i, t) = M_i(e_x^{i-1}, t) + N_i(e_x^{i-1}, t) e_{x_i} \\ i = 2, \ldots, r; \quad e_x^i = \operatorname{col}(e_{x_l}, \ldots, e_{x_i}), \quad e_x^r = e_x, \quad M_1 = 0, \quad N = I_m \\ M_i(e_x^{i-1}, t) = -L_i^{-1}(\Phi^{i-1}, t) K_i(\Phi^{i-1}, t), \quad N_i(e_x^{i-1}, t) = L_i^{-1}(\Phi^{i-1}, t) \\ \Phi_{r+1}(e_x, e_w, t) = M_{r+1}(e_x, t) + N_{r+1}(e_x, t) e_w \\ M_{r+1}(e_x, t) = -L_{r+1}^{-1}(\Phi^r, t) K_{r+1}(\Phi^r, t), \quad N_{r+1}(e_x, t) = L_{r+1}^{-1}(\Phi^r, t) \\ \Phi^{i-1} = \operatorname{col}(\Phi_1, \ldots, \Phi_{i-1}), \quad \Phi^{i-1} = \Phi^{i-1}(e_x^{i-1}, t) \\ \Phi_k = \Phi_k(e_x^k, t), \quad k = 1, \ldots, i-1; \quad \Phi^r = \Phi^r(e_x^r, t)\end{gathered} \tag{3.15}$$

We can similarly construct one-to-one transformations for the canonical forms (2.1), (2.2), (2.7), (2.4)–(2.6); (2.1), (2.2), (2.8), (2.4)–(2.6) and (2.9)–(2.10). They can also be obtained from (3.1)–(3.5), (3.9), (3.10), (3.14) and (3.15) by replacing (2.3) by (2.7) and (2.8), respectively, or by replacing (2.3) and (2.4) by the matrices P and Q from (2.9). In Section 7 we will derive explicit formulae (7.2) and (7.3) for the direct and inverse transformations of (3.1)–(3.5), (3.14) and (3.15) for the electromechanical reversible controlled systems considered in [4–14].

4. CRITERIA FOR THE STABILITY AND STABILIZATION OF PROGRAMMED MOTION FOR REVERSIBLE CONTROLLED SYSTEMS WITH A CANONICAL FORM OF THE FIRST TYPE

We will first consider the problem of ensuring asymptotic stability of the programmed motion in a closed reversible controlled system, whose dynamics can be represented in the canonical form of the first type (2.1)–(2.8). We will synthesize the stabilizing control law with feedback with respect to e_x in the form

$$e_w = \Gamma_0(e_x, t) e_x \tag{4.1}$$

where the matrix of the gains

$$\Gamma_0(e_x, t) = \| \Gamma_{01}(e_x, t), \ldots, \Gamma_{0r}(e_x, t) \| \tag{4.2}$$

is an $m \times n$ matrix function consisting of $m \times n$ blocks Γ_{0j} $(j = 1, \ldots, r)$.

Suppose the reversible controlled system has a canonical form of the first subclass of the first type, i.e. it is described by Eqs (2.1)–(2.2), (2.7) and (2.4)–(2.6). We will construct the matrix Γ_0 given by (4.2) in the form

$$\Gamma_0(e_x,t) = \|0\Gamma_{0r}(e_x,t)\| \tag{4.3}$$

$$\Gamma_{0r}(e_x,t) = Q_r^{-1}(e_x^r,t)\overline{\Gamma}_{0r}(e_x^{r-1},t) \tag{4.4}$$

so that

$$\Gamma(e_x^{r-1},t) = P(e_x^{r-1},t) + Q(e_x^r,t)\Gamma_0(e_x,t) \tag{4.5}$$

and we will choose the blocks P_{ii} $(i = 1, \ldots, r)$ of the matrix function P (2.7) and (2.5) and the block Γ_{0r} of (4.4), where Γ_{0r} is an $m \times m$ matrix function, in the matrix function Γ_0 (4.3), so that the matrix function Γ (4.5) is such that

$$-[\Gamma(e_x^{r-1},t) + \Gamma^*(e_x^{r-1},t)]/2 = G_1(e_x^{r-1},t) \tag{4.6}$$

Here G_1 is a quasi-diagonal symmetric matrix function of the form

$$G_1(e_x^{r-1},t) = \operatorname{diag}(G_{11}(t), \quad G_{12}(e_x^1,t),\ldots,G_{1r}(e_x^{r-1},t)) \tag{4.7}$$

where

$$\begin{gathered} G_{11}(t) = -(P_{11}(t) + P_{11}^*(t))/2, \quad G_{1i}(e_x^{i-1},t) = -(P_{ii}(e_x^{i-1},t) + P_{ii}^*(e_x^{i-1},t))/2, \\ i = 2,\ldots,r-1 \\ G_{1r}(e_x^{r-1},t) = -(P_{rr}(e_z^{r-1},t) + \overline{\Gamma}_{0r}(e_x^{r-1},t) + P_{rr}^*(e_x^{r-1},t) + \overline{\Gamma}_{0r}^*(e_x^{r-1},t))/2 \end{gathered} \tag{4.8}$$

are $m \times m$ blocks with positive diagonal elements with a predominant diagonal, i.e. the following inequalities are satisfied for their elements g_{1ikj} $(k, j = 1, \ldots, m; i = 1, \ldots, r)$

$$\begin{gathered} g_{11kk}(t) > 0, \quad t \geqslant t_0, \quad k = 1\ldots,m; \quad g_{1ikk}(e_x^{i-1},t) > 0, \quad \forall e_x^{i-1} \in R^{m(i-1)}, \\ t \geqslant t_0, \quad k = 1,\ldots,m \quad i = 2,\ldots,r \\ \inf_{t \geqslant t_0}\left[g_{11kk}(t) - \sum_{\substack{j=1 \\ j\neq k}}^m |g_{11kj}(t)|\right] \geqslant \delta_{1k} > 0, \quad t \geqslant t_0, \quad k = 1,\ldots,m \\ \inf_{e_x^{i-1} \in R^{m(i-1)}, t \geqslant t_0}\left[g_{1ikk}(e_x^{i-1},t) - \sum_{\substack{j=1 \\ j\neq k}}^m |g_{1ikj}(e_x^{i-1},t)|\right] \geqslant \delta_{ik} > 0, \quad k = 1,\ldots,m; \quad i = 2,\ldots,r \end{gathered} \tag{4.9}$$

where δ_{ij} $(k = 1, \ldots, m; i = 1, \ldots, r)$ are certain positive constants.

Then the equation of the transients in the closed canonical reversible controlled system (2.1), (2.2), (2.7), (2.4)–(2.6) and (4.1)–(4.9) has the form

$$\dot{e}_x = \Gamma(e_x^{r-1},t)e_x, \quad e_x(t_0) = e_{x_0}, \quad t \geqslant t_0 \tag{4.10}$$

where Γ is the matrix function of (4.5)–(4.9).

We will consider the function

$$V(e_x) = \tfrac{1}{2}|e_x|^2 \tag{4.11}$$

and calculate its derivative with respect to t by virtue of (4.10) and (4.5)–(4.9)

$$V^{\cdot}(e_x(t)) = \tfrac{1}{2}(|e_x(t)|^2)^{\cdot} = \tfrac{1}{2}e_x^*(t)(\Gamma(e_x^{r-1},t)+\Gamma^*(e_x^{r-1},t))e_x(t)) = \\ = -e_x^*(t)G_1(e_x^{r-1},t)e_x(t), \quad t \geq t_0 \tag{4.12}$$

It follows from (4.9) that the quasi-diagonal symmetric matrix function G_1 (4.7)–(4.9) is positive definite for all values of its arguments, i.e.

$$G_1(e_x^{r-1},t) > 0, \quad \forall e_x^{r-1} \in R^{m(r-1)}, \quad t \geq t_0 \tag{4.13}$$

Hence we obtain the following limit from (4.12) and (4.11)

$$V^{\cdot}(e_x(t)) = -e_x^*(t)G_1(e_x^{r-1},t)e_x(t) \leq -\gamma_1|e_x(t)|^2 = -2\gamma_1 V(e_x(t)), \quad t \geq t_0 \tag{4.14}$$

Here γ_1 is a parameter such that

$$0 < \gamma_1 \leq \inf_{e_x^{r-1} \in R^{m(r-1)}, t \geq t_0} \lambda_m(e_x^{r-1},t) \tag{4.15}$$

where λ_m (e_x^{r-1}, t) is the minimum eigenvalue of the positive definite matrix function G_1 of (4.7)–(4.9) and (4.13). From (4.14) and (4.15) we obtain $c(e_x(t)) \leq V(e_x(t_0)) \exp[-2\gamma_1(t-t_0)]$, $t \geq t_0$. Hence, once again using (4.11) we obtain

$$|e_x(t)|^2 \leq |e_{x_0}|^2 \exp[-2\gamma_1(t-t_0)], \quad e_x(t_0) = e_{x_0}, \quad t \geq t_0 \tag{4.16}$$

Consequently, the position of equilibrium $e_x = 0$ of system (4.10), (4.5)–(4.9) is asymptotically stable as a whole with the limit

$$|e_x(t)| \leq |e_{x_0}| \exp[-\gamma_1(t-t_0)], \quad e_x(t_0) = e_{x_0}, \quad t \geq t_0 \tag{4.17}$$

Substituting e_x from (3.1) and e_w from (4.1)–(4.9) into (3.14) and (3.15) we obtain the stabilizing control law with feedback with respect to e_z

$$e_u = \Phi_{r+1}(e_x, \Gamma_0(e_x,t)e_x,t) = \Phi_{r+1}[\Psi^r(e_z,t), \Gamma_0(\Psi^r(e_z,t),t)\Psi^r(e_z,t),t] \tag{4.18}$$

for the initial reversible controlled system (1.3)–(1.9), (1.12) and (1.13).

The equation of the transient in the closed initial reversible controlled system (1.3)–(1.9), (1.12), (1.13), (4.18) and (4.3)–(4.9) has the form

$$e_z^0 = F_z(e_z, \Phi_{r+1}(\Psi^r(e_z,t), \Gamma_0(\Psi^r(e_z,t),t)\Psi^r(e_z,t),t),t), \quad e_z(t_0) = e_{z_0}, \quad t \geq t_0 \tag{4.19}$$

Using the relations for the finite increments of the vector function $\Phi^r(e_x, t)$ (3.14) and (3.15) and for the limit of the transient (4.17), (1.12) and (1.3), we obtain for the canonical reversible controlled system

$$|e_z(t)| = |\Phi^r(e_x,t)| = \left|\left(\int_0^1 J_{\Phi^r}(\theta e_x^{r-1},t)d\theta)e_x(t)\right)\right| \leq \mu_0|e_x(t)| \leq \\ \leq \mu_0|e_x(t_0)|\exp[-\gamma_1(t-t_0)] = \mu_0|\Psi^r(e_z(t_0),t_0)|\exp[-\gamma_1(t-t_0)], \quad t \geq t_0 \tag{4.20}$$

Here

$$\sup_{\tilde{e}_x^{r-1} \in [0, e_x^{r-1}], t \geq t_0} |J_{\Phi^r}(\tilde{e}_x^{r-1},t)| \leq \nu_0 + \sum_{j=1}^{s} \nu_j |\tilde{e}_x^{r-1}|^j = \mu_0 \\ J_{\Phi^r}(e_x^{r-1},t) = \partial\Phi^r(e_x,t)/\partial e_x, \quad [0, e_x^{r-1}] = \{\tilde{e}_x^{r-1} | \tilde{e}_x^{r-1} = \theta e_x^{r-1}, 0 \leq \theta \leq 1\} \tag{4.21}$$

and $\nu_0 > 0$, $\nu_j \geqslant 0$ ($j = 1, \ldots, s$) are certain parameters.

It follows from (4.20) and (4.21) that the position of equilibrium $e_z = 0$ of the initial reversible controlled system (1.3)–(1.9), (1.12) and (1.13) with control law e_u (4.18) and (4.3)–(4.9) is asymptotically stable as a whole. Consequently, the programmed motion $z_p(t)$ of the initial reversible controlled system (1.1), (1.3)–(1.9), (1.12) and (1.13) also with control law (1.13), (4.18) and (4.3)–(4.9) is asymptotically stable as a whole with the transient limit (4.20) and (4.21).

We will now consider the problems of the stability of the programmed motion and of synthesizing a stabilizing control law for the reversible controlled system with a canonical form of the second subclass of the first type (2.1)–(2.2), (2.8) and (2.4)–(2.6).

For the canonical reversible controlled system (2.1)–(2.2), (2.8) and (2.4)–(2.6) we will synthesize a stabilizing control law e_w with feedback with respect to e_x in the form (4.1), (4.3) and (4.4) and we will choose the blocks P_{ii} ($i = 1, \ldots, r-1$) of the matrix function P (2.8) and (2.5) and the block Γ_{0r} (4.4) of the matrix function Γ_0 (4.3) so that the matrix function Γ (4.5), (2.8) and (4.3) is such that

$$-[\Gamma(e_x^{r-1},t)+\Gamma^*(e_x^{r-1},t)]/2 = G_1(e_x^{r-1},t)+G_2(e_x^{r-2},t) \tag{4.22}$$

Here G_1 is a quasi-diagonal symmetric positive definite matrix function (4.7)–(4.9), (4.13) and the matrix function G_2 has the form

$$G_2(e_x^{r-2},t) = -\frac{1}{2}\begin{Vmatrix} 0 & P_{12}(t) & 0 & & 0 \\ P_{12}^*(t) & 0 & P_{23}(e_x^1,t) & 0 & 0 \\ 0 & P_{23}^*(e_x^1,t) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & P_{r-1,r}(e_x^{r-2},t) \\ 0 \;\ldots & & 0 & P_{r-1,r}^*(e_x^{r-2},t) & 0 \end{Vmatrix} \tag{4.23}$$

Suppose the following relations are satisfied

$$\begin{gathered}
\inf_{t\geqslant t_0}\left[\delta_1 - \frac{1}{2}\varepsilon|P_{12}(t)|\right] \geqslant \alpha_1 > 0 \\
\inf_{e_x^1\in R^m,\, t\geqslant t_0}\left[\delta_2 - \frac{1}{2}\varepsilon|P_{12}(t)| - \frac{1}{2}\varepsilon|P_{23}(e_x^1,t)|\right] \geqslant \alpha_2 > 0 \\
\inf_{e_x^{i-1}\in R^{m(i-1)},\, t\geqslant t_0}\left[\delta_i - \frac{1}{2}\varepsilon|P_{i-1,i}(e_x^{i-2},t)| - \frac{1}{2}\varepsilon|P_{i,i+1}(e_x^{i-1},t)|\right] \geqslant \alpha_i > 0, \quad i = 3,\ldots,r-1 \\
\inf_{e_x^{i-2}\in R^{m(r-2)},\, t\geqslant t_0}\left[\delta_r - \frac{1}{2}\varepsilon|P_{r-1,r}(e_x^{r-2},t)|\right] \geqslant \alpha_r > 0
\end{gathered} \tag{4.24}$$

where $\delta_i = \min_{k=1,\ldots,m}\delta_{ik}$, δ_{ik} ($k = 1, \ldots, m$; $i = 1, \ldots, r$) are positive constants from (4.9) and ε, α_i ($i = 1, \ldots, r$) are certain positive parameters.

The equation of the transient in the closed canonical reversible controlled system (2.1), (2.2), (2.8), (2.4)–(2.6), (4.1)–(4.5), (4.7)–(4.9) and (4.22)–(4.24) then has the form

$$e_x^{\cdot} = \Gamma(e_x^{r-1},t), e_x, \quad e_x(t_0) = e_{x_0}, \quad t \geqslant t_0 \tag{4.25}$$

Making the following replacement of variable in (4.25)

$$e_x = He_y \tag{4.26}$$

where $e_y = \operatorname{col}(e_{y_1}, \ldots, e_{y_r})$, e_{y_i} ($i = 1, \ldots, r$) are n- and m-dimensional vectors

$$H = \operatorname{diag}(I_m, \varepsilon I_m, \ldots, \varepsilon^{r-1} I_m) \tag{4.27}$$

is a constant, diagonal, positive definite $n \times n$ matrix, and $\varepsilon > 0$ is the parameter from (4.24), we can reduce it to the system

$$e_y^{\cdot} = \Gamma_y(e_y^{r-1},t)e_y, \quad e_y(t_0) = e_{y_0}, \quad t \geqslant t_0 \tag{4.28}$$

Here

$$\Gamma_y(e_y^{r-1},t)=H^{-1}\Gamma(H_1e_y^{r-1},t)H \tag{4.29}$$

where $H_1=\mathrm{diag}(I_m,\varepsilon I_m,\dots,\varepsilon^{r-2}I_m)$ and $\varepsilon>0$ is the parameter from (4.24), where the matrix Γ_y (4.29) is such that

$$-\tfrac{1}{2}(\Gamma_y(e_y^{r-1},t)+\Gamma_y^*(e_y^{r-1},t))=-\tfrac{1}{2}(H^{-1}\Gamma(H_1e_y^{r-1},t)H+H\Gamma^*(H_1e_y^{r-1},t)H^{-1})=$$
$$=-\tfrac{1}{2}(H^{-1}\Gamma(e_x^{r-1},t)H+H\Gamma^*(e_x^{r-1},t)H^{-1})=G_1(e_x^{r-1},t)+\varepsilon G_2(e_x^{r-2},t) \tag{4.30}$$

where G_1 is the positive definite matrix function (4.7)–(4.9) and (4.13) while G_2 is the matrix function (4.23) which satisfied (4.24).

Consider the function

$$V(e_y)=\tfrac{1}{2}|e_y|^2 \tag{4.31}$$

We will calculate its derivative with respect to t by virtue of (4.28), taking into account relations (4.29)–(4.31), (4.7)–(4.9), (4.13) and (4.22)–(4.24). We obtain the relation

$$V^{\cdot}(e_y(t))=\tfrac{1}{2}(|e_y(t)|^2)^{\cdot}=\tfrac{1}{2}e_y^*(t)(\Gamma_y(e_y^{r-1},t)+\Gamma_y^*(e_y^{r-1},t))e_y(t)=$$
$$=-e_y^*(t)(G_1(H_1e_y^{r-1},t)+\varepsilon G_2(H_2e_y^{r-2},t))e_y^*(t)=$$
$$=-e_y^*(t)(G_1(e_x^{r-1},t)+\varepsilon G_2(e_x^{r-2},t))e_y(t)\leqslant-\sum_{i=1}^{r}\alpha_i|e_y(t)|^2\leqslant-\gamma_2V(e_y(t)),\ \ t\geqslant t_0 \tag{4.32}$$

where $H_2=\mathrm{diag}(I_m,\varepsilon I_m,\dots,\varepsilon^{r-3}I_m)$; $\varepsilon>0$, $\alpha_i>0$ $(i=1,\dots,r)$ are parameters from (4.24) and $\gamma_2=\min_i\alpha_i$, $i=1,\dots,r$.

From (4.32) we find $V(e_y(t))\geqslant V(e_y(t_0))\exp[-2\gamma_2(t-t_0)]$, $t\geqslant t_0$. Hence, using (4.31) once gain we obtain the relation

$$|e_y(t)|^2\leqslant|e_{y_0}|^2\exp[-2\gamma_2(t-t_0)],\ \ e_y(t_0)=e_{y_0},\ \ t\geqslant t_0 \tag{4.33}$$

Consequently, the position of equilibrium $e_y=0$ of system (4.28) is asymptotically stable as a whole with the limit

$$|e_y(t)|\leqslant|e_{y_0}|\exp[-\gamma_2(t-t_0)],\ \ e_y(t_0)=e_{y_0},\ \ t\geqslant t_0 \tag{4.34}$$

Hence it also follows from (4.26) and (4.27) that the position of equilibrium $e_x=0$ of system (4.25), (4.5), (4.22)–(4.24), (4.7)–(4.9) and (4.13) is asymptotically stable as a whole with the limit

$$|e_x(t)|\leqslant\beta_1|e_{x_0}|\exp[-\gamma_2(t-t_0)],\ \ e_x(t_0)=e_{x_0},\ \ t\geqslant t_0 \tag{4.35}$$

where $\beta_1=|H|\ |H^{-1}|$.

By analogy with the preceding case, using (4.20), (4.21) and (4.35) it can be shown that the position of equilibrium $e_z=0$ of the closed initial reversible controlled system (4.19), (1.4)–(1.9), (1.12), (1.13), (4.3)–(4.5), (4.22)–(4.24), (4.7)–(4.9) and (4.13) is asymptotically stable as a whole and, consequently, the programmed motion $z_p(t)$ of the initial reversible controlled system (1.1), (1.3)–(1.9), (1.12) and (1.13) with control law u (1.14), (4.18), (4.3)–(4.5), (4.22)–(4.24), (4.7)–(4.9) and (4.13) is also asymptotically stable as a whole with transient limit

$$|e_z(t)|=|\Phi^r(e_x,t)\leqslant\mu_0|e_x(t)|\leqslant\mu_0\beta_1|e_x(t_0)|\exp[-\gamma_2(t-t_0)]\leqslant$$
$$\leqslant\mu_0\beta_1|\Psi^r(e_z(t_0),t_0)|\exp[-\gamma_2(t-t_0)],\ \ t\geqslant t_0 \tag{4.36}$$

where μ_0 is a positive parameter defined in the same way as (4.21) and γ_2 and β_1 are the parameters from (4.35).

5. CRITERIA OF THE STABILITY AND STABILIZATION OF THE PROGRAMMED OPTION FOR A REVERSIBLE CONTROLLED SYSTEM WITH CANONICAL FORM OF THE SECOND TYPE

We will now consider the problems of the stability and stabilization of the programmed motion for a reversible controlled system which can be represented in canonical form of the second type (2.9), (2.2) and (2.10). Taking into account the structure of the constant matrices P and Q from (2.9) and (2.10) it can be shown that rank $||Q, PQ, \ldots, P^{r-1}Q||=n$. Consequently, the pair $(P, Q|$ is controllable and a constant $m\times m$ matrix

$$\Gamma_0=||\Gamma_{01},\ldots,\Gamma_{0r}|| \tag{5.1}$$

exists, where Γ_{0j} $(j=1,\ldots,r)$ are $m\times m$ blocks such that the matrix

$$\Gamma=P+Q\Gamma_0 \tag{5.2}$$

$$\operatorname{Re}\lambda_i(\Gamma)<0,\quad i=1,\ldots,n \tag{5.3}$$

where $\lambda_i(\Gamma)$ $(i=1,\ldots,n)$ are the eigenvalues of Γ.

We will synthesize the control law with feedback with respect to e_x in the form

$$e_w=\Gamma_0 e_x \tag{5.4}$$

Then the equation of the transient in the closed canonical reversible controlled system (2.9), (2.2), (2.10), (5.4) and (5.1)–(5.3) has the form

$$e_x^{\cdot}=\Gamma e_x,\quad e_x(t_0)=e_{x0},\quad t\geqslant t_0 \tag{5.5}$$

Consequently, the position of equilibrium $e_x=0$ in the closed reversible controlled system (5.5), (5.2) and (5.3) is asymptotically stable as a whole with transient limit of the form

$$|e_x(t)|\leqslant\beta_2|e_x(t_0)|\exp[\gamma_3(t-t_0)],\quad t\geqslant t_0 \tag{5.6}$$

where $\gamma_3=\max_i \operatorname{Re}\lambda_i(\Gamma)$ $(i=1,\ldots,n)$, and $\beta_2>0$ is a parameter which depends solely on the Γ.

Substituting e_x from (3.1) and e_w from (5.4) and (5.1)–(5.3) into (3.14) and (3.15) we obtain the stabilizing control law with feedback with respect to e_z

$$e_u=\Phi_{r+1}(e_x,\Gamma_0 e_x,t)=\Phi_{r+1}(\Psi^r(e_z,t),\Gamma_0\Psi^r(e_z,t),t) \tag{5.7}$$

and the equation of the transient in the closed initial reversible controlled system (1.3)–(1.9), (1.12), (1.13), (5.7) and (5.1)–(5.3) of the form

$$e_z^{\cdot}=F_z(e_z,\Phi_{r+1}(\Psi^r(e_z,t),\ \Gamma_0\Psi^r(e_z,t),t),t),\quad e_z(t_0)=e_{z0},\quad t\geqslant t_0 \tag{5.8}$$

Estimating the transient in (5.8) using (4.20) and (5.6) we obtain

$$\begin{aligned}|e_z(t)|&=|\Phi^r(e_x,t)|\leqslant\mu_0|e_x(t)|\leqslant\mu_0\beta_2|e_x(t_0)|\exp[\gamma_3(t-t_0)]=\\&=\mu_0\beta_2|\Psi^r(e_z(t_0),t_0)|\exp[\gamma_3(t-t_0)],\quad t\geqslant t_0\end{aligned} \tag{5.9}$$

where μ_0 is a parameter defined in the same way as (4.21), and β_2 and γ_3 are the parameters from (5.6).

Hence it follows that the position of equilibrium $e_z=0$ in the closed initial reversible controlled system (5.8), (1.4)–(1.9), (1.12), (1.13), (5.7) and (5.1)–(5.3) is asymptotically stable as a whole. Consequently, the programmed motion $z_p(t)$ of the initial reversible controlled system (1.1), (1.4)–(1.9), (1.12) and (1.13) with control law (1.14), (5.7) and (5.1)–(5.3) is asymptotically stable as a whole with the transient limit (5.9).

6. DECOMPOSABILITY, ROBUSTNESS AND STABILIZATION QUALITY OF THE REVERSIBLE CONTROLLED SYSTEM

Crossed dynamic couplings are an inherent feature of multidimensional reversible controlled systems. The intensity of the interaction between these couplings depends non-linearly on the actual state. Because of this, when using linear proportional-integral controllers in a closed reversible controlled system the quality of the transient deteriorates considerably and a loss of stability of the programmed motion becomes possible.

An advantage of the synthesized non-linear stabilization laws of programmed motion is the fact that, by a correct choice of the parameters of the matrices of the gains, one can ensure complete compensation of the crossed couplings and a specified form of the attenuation of the transients in a closed reversible controlled system.

A reversible controlled system will be said to be decomposable if a control law exists for which the equation of the transient in the closed system can be expanded into a system of independent equations in the controlled coordinates.

It can be shown that synthesized control laws ensure that a reversible controlled system is decomposable in this sense. Thus, for example, for a reversible controlled system in canonical form of the second type (2.9), (2.2) and (2.10) it is sufficient to choose the blocks of the matrix P from (2.9) and of the matrix Γ_0 (5.1) such that

$$P_{ij} = \operatorname{diag}(P_{ijkk})_{k=1}^{m}, \quad i=1,\dots,r-1; \quad j=1,\dots,i+1 \tag{6.1}$$

$$\Gamma_{0j} = Q_r^{-1}(-P_{rj}+\hat{\Gamma}_{0j}), \quad \hat{\Gamma}_{0j} = \operatorname{diag}(\hat{\Gamma}_{0jkk})_{k=1}^{m}, \quad j=1,\dots,r$$

Then, the equation of the transient (5.5) can be split into m independent equations of the form

$$\dot{\bar{e}}_{x_k} = \bar{\Gamma}_k \bar{e}_{x_k}, \quad \bar{e}_{x_k}(t_0) = e_{x_{k0}}, \quad t \geq t_0, \quad k=1,\dots,m \tag{6.2}$$

$$\bar{\Gamma}_k = \begin{Vmatrix} p_{11kk} & p_{12kk} & 0 & & \cdots & 0 \\ p_{21kk} & p_{22kk} & p_{23kk} & 0 & \cdots & 0 \\ \vdots & & \cdot & & \ddots & \vdots \\ p_{r-1,1kk} & \cdots & & & & p_{r-1,rkk} \\ \hat{\Gamma}_{01kk} & \cdots & & & & \hat{\Gamma}_{0rkk} \end{Vmatrix}$$

where $\bar{e}_{x_k} = \operatorname{col}(e_{x_{1k}}, \dots, e_{x_{rk}})$ is an r-dimensional vector of the state of the system, where $e_x = \operatorname{col}(e_{x_1}, \dots, e_{x_k})$ are n- and m-vectors.

The parameters of the equations of the transient (6.2), taking (6.1) into account, are the coefficients $\hat{\Gamma}_{0jkk}$ ($j = 1, \dots, r$; $k = 1, \dots, m$)—the elements of the matrix $\hat{\Gamma}_0 = \|\hat{\Gamma}_{01}, \dots, \hat{\Gamma}_{0r}\|$, where $\hat{\Gamma}_{0j} = Q_r\Gamma_{0j} + P_{rj} = \operatorname{diag}(\hat{\Gamma}_{0jkk})_{k=1}^{m}$ ($j = 1, \dots, m$) found from the matrix of the gains Γ_0 (5.1) of the stabilizing control law e_w (5.4) and (5.1)–(5.3).

It is obvious that they can always be chosen so that the transient has a previously specified form and rate of damping. In the important practical case when $n = 3m$, which corresponds to the dynamics of electromechanical reversible controlled systems, the parameters of the law of stabilization of the programmed motion can be chosen using Vyshnegradskii diagrams, starting from the requirement to ensure the desired form of transient damping (monotonic, aperiodic or oscillatory) and specified figures of merit (accuracy, speed of response, etc.).

Theoretical formulae have been obtained [4, 6] for the parametric synthesis of non-linear stabilizing and modal laws of control of reversible controlled systems in a canonical form of the second type with Vyshnegradskii parameters. It is important to note that these stabilization laws ensure robustness of the reversible controlled system, i.e. the stability of the programmed motion with respect to limited parametric and constantly acting perturbations [4, 6]. Using non-linear transformations in state space and the equations proposed above, it is easy to extend these results to a wider class of reversible controlled systems and to synthesize corresponding robust stabilization laws of the programmed motion with feedback with respect to the initial or canonical state vectors. On the basis of the stabilizing and robust stabilization laws one can synthesize adaptive control of the programmed motion of reversible controlled systems using the methods described in [1–3, 7–14].

7. APPENDIX

For electromechanical reversible controlled systems with a d.c. motor with a rigid reduction gear, which describes the non-linear dynamics of robots, lathes, etc. [4–14], the equations in deviations have the form (1.3)–(1.9), where $n = 3m$ and

$$\begin{aligned}
&e_z = \mathrm{col}(q-q_p, q^\cdot - q_p^\cdot, I - I_p),\quad F_{z_1}(e_z^2,t) = q^\cdot - q_p^\cdot, F_{z_2}(e_z^3,t) = \\
&= A^{-1}(q)(k_M I - b(q,q^\cdot,t)) - A^{-1}(q_p)(k_M I_p - b(q_p,q_p^\cdot,t)),\quad F_{z_3}(e_z^3,e_u,t) = \\
&= L^{-1}(e_u - R(I-I_p) - k_e i_p(q^\cdot - q_p^\cdot)),\quad C_1(r_z^1,t) = 0,\quad D_1(e_z^1,t) = I_m \\
&C_2(e_z^2,t) = A^{-1}(q)(k_M I_p - b(q,q^\cdot,t)) - A^{-1}(q_p)(k_M I_p - b(q_p q_p^\cdot,t)),\quad D_2(e_z^2,t) = A^{-1}(q)k_M \\
&C_3(e_z^3,t) = -L^{-1}(R(I-I_p) - k_e i_p(q^\cdot - q_p^\cdot)),\quad D_3(e_z^3,t) = L^{-1} \\
&A(q) = Ji_p + i_p^{-1}A_0(q),\quad b(q,q^\cdot,t) = k_0 i_p q^\cdot + i_p^{-1} b_0(q,q^\cdot,t)
\end{aligned} \tag{7.1}$$

Here q is an m-dimensional vector of the generalized coordinates of the mechanical part—the slave mechanism of the electromechanical reversible controlled system, $A_0(q)$ is a positive definite $m \times m$ matrix of the kinetic energy $T = \frac{1}{2}q^{\cdot *}A_0(q)q$ of the slave mechanism of the electromechanical reversible controlled system

$$b_0(q,q^\cdot,t) = A_0^\cdot(q)q^\cdot - \tfrac{1}{2}\partial(q^{\cdot *}A_0(q)q^\cdot)/\partial q + \partial\Pi/\partial q - Q_0;$$

$\Pi = \Pi(q)$ is the potential energy of the slave mechanism of the reversible controlled system, $Q_0 = Q_0(q, q^\cdot, t)$ is an m-dimensional vector of the generalized forces (torques) of the resistance acting on the slave mechanism, I is an m-dimensional vector of the currents in the armature circuits of the d.c. motor, e_u is an m-dimensional vector of the controls—the deviations of the controlling voltages u, applied to the armature circuits of the d.c. motor, from their programmed values u_p, J, k_0, k_M, L, R, k_e are diagonal matrices of the electromechanical parameters of the d.c. motor, which are positive real quantities, and i_p is a diagonal matrix of the transfer constants of the reduction gears (such that $\varphi = i_p q$, where φ is the vector of the angles of rotation of the shafts of the motors).

For the electromechanical reversible controlled system (1.3)–(1.9), (7.1) the operators of direct and inverse transformations in state spaces and Eqs (3.1) and (3.14) have the following respective forms

$$\begin{aligned}
&\Psi^3(e_z,t) = \left\| \begin{matrix} e_{z_1} \\ K_2(e_z^1,t) + L_2(e_z^1,t)e_{z_2} \\ K_3(e_z^2,t) + L_3(e_z^2,t)e_{z_3} \end{matrix} \right\|,\quad \Phi^3(e_x,t) = \left\| \begin{matrix} e_{x_1} \\ M_2(e_x^1,t) + N_2(e_x^1,t)e_{x_2} \\ M_3(e_x^2,t) + N_3(e_x^2,t)e_{x_3} \end{matrix} \right\| \\
&\Psi_4(e_z,e_u,t) = K_4(e_z,t) + L_4(e_z,t)e_u,\quad \Phi_4(e_x,e_w,t) = M_4(e_x,t) + N_4(e_x,t)e_w
\end{aligned} \tag{7.2}$$

Here

$$K_2(e_z^1,t) = -P_{12}^{-1}(e_z^1,t)P_{11}(e_z^1,t),\quad L_2(e_z^1,t) = P_{12}^{-1}(e_z^1,t)$$

$$K_3(e_z^2,t) = P_{23}^{-1}(\Psi^2,t)\left[-\sum_{k=1}^{2} P_{2k}(\Psi^2,t)\Psi_k + K_2^\cdot(e_z^1,t) + L_2^\cdot(e_z^1,t)e_{z_2} + L_2(e_z^1,t)C_2(e_z^2,t)\right]$$

$$L_3(e_z^2,t)=P_{23}^{-1}(\Psi^2,t)L_2(e_z^1,t)D_2(e_z^2,t) \tag{7.3}$$

$$K_4(e_z,t)=P_{34}^{-1}(\Psi^3,t)\left[-\sum_{k=1}^{3}P_{3k}(\Psi^3,t)\Psi_k+\dot{K}_3(e_z^2,t)+\dot{L}_3(e_z^2,t)e_{z_3}+L_3(e_z^2,t)C_3(e_z^3,t)\right]$$

$$L_4(e_z,t)=P_{34}^{-1}(\Psi^3,t)L_3(e_z^2,t)D_3(e_z^3,t)$$

where the matrix functions P_{ij} ($i = 1, \ldots, 3; j = 1, \ldots, i + 1$) and the vector functions Ψ_k, Ψ^k ($k = 1, 2, 3$) are defined in (2.3), (2.5) and (3.10), while the vector functions M_i ($i = 2, 3, 4$) and the matrix functions N_i ($i = 2, 3, 4$) are defined by (3.15).

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